Proposition: Let R be an integral domain such that every prime ideal is maximal. Then R is a field.

Pf. Notice that $\langle 0 \rangle = \{0\}$ is prime because if $a \cdot b \in \langle 0 \rangle \iff a \cdot b = 0 \overset{}{\underset{\text{I.D.}}{\iff}} a=0$ or $b=0$

$\Rightarrow \langle 0 \rangle$ is prime. By the given, it's maximal.

$\Rightarrow R/\langle 0 \rangle$ is a field, so $R \cong R/\langle 0 \rangle$ is a field.

Examples Consider $\mathbb{Z}[i] = \{a+bi : a,b \in \mathbb{Z}\}$.

(by defn, $\mathbb{Z}[i] = \{a_0 + a_1 i + a_2 i^2 + \cdots + a_n i^k : a_j \in \mathbb{Z}\ \forall j\}$
$= \{a+bi : a,b \in \mathbb{Z}\}$.)

What is $\mathbb{Z}[i]/\langle 2-i \rangle$ isomorphic to?

What are distinct elements of $\mathbb{Z}[i]/\langle 2-i \rangle$?

$$\langle 2-i \rangle = \{(2-i)(a+bi) : a,b \in \mathbb{Z}\}$$
$$= \{(2a+b) + i(2b-a) : a,b \in \mathbb{Z}\}$$

$$\left\{\begin{array}{l} 0 + \langle 2-i \rangle \\ 1 + \langle 2-i \rangle \\ 2 + \langle 2-i \rangle \\ 3 + \langle 2-i \rangle \\ 4 + \langle 2-i \rangle \end{array}\right.$$

must be distinct elements of $\langle 2-i \rangle$

Let $r + si = (2a+b) + i(2b-a)$

what can r,s be?

$$\begin{pmatrix} r \\ s \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}}_{\text{what is the image on } \mathbb{Z}?} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}$$

$$\mathbb{Z} \oplus \mathbb{Z} \ni \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{5}(2r-s) \\ \frac{1}{5}(r+2s) \end{pmatrix}$$

$$\Rightarrow 2r-s \equiv 0 \bmod 5$$
$$r+2s \equiv 0 \bmod 5$$
$$\Rightarrow 3r+s \equiv 0 \bmod 5$$

if $s=0$ r must be a multiple of 5

Note $i = 2-(2-i)$

$[i]=[2]$ in $\mathbb{Z}[i]/\langle 2-i\rangle \Rightarrow [2i]=[4]$

Any integer is in one of these classes, because

$5 \in \langle 2-i\rangle$.

$\therefore$ any element of $\mathbb{Z}[i]$ is in one of these 4 classes.

$$[a+bi] = [(a+b\cdot 2) \bmod 5].$$

$\therefore$ $\mathbb{Z}[i]/\langle 2-i\rangle \cong \mathbb{Z}_5$, a field, so

$\langle 2-i\rangle$ is a maximal ideal.

(and prime!)

Example: Consider $\mathbb{R}[x]/\langle x^2+1\rangle$. What is this isomorphic to?

$$\langle x^2+1\rangle = \{(x^2+1)p(x) : p(x) \text{ is a poly. in } \mathbb{R}[x].\}$$
$$= \{q(x) \in \mathbb{R}[x] : q(i)=q(-i)=0\}$$

[Note $q(x)=0 \iff \overline{q(x)}=\overline{0}=0$

$\iff q(\overline{x})=0$

$$= \{q(x) \in \mathbb{R}[x] : q(i)=0\}.$$

Note $(x^2+1)p(x)$ is always at least degree 2.

$\Rightarrow$ $a+bx+\langle x^2+1\rangle$ are distinct elements of $\mathbb{R}[x]/\langle x^2+1\rangle$

for $a, b \in \mathbb{R}$.

For any polynomial $p(x)$ of degree $\geqslant 2$ with real coeff's, we ask if it is possible that

$$p(x) \neq a + bx + c(x)(x^2+1) \quad \text{for some } a, b \in \mathbb{R}.$$

Answer: No! In fact, for any poly. $p(x)$ of degree ≥ 2, $\exists$ $a, b \in \mathbb{R}$ and a $q(x) \in \mathbb{R}[x]$ s.t.

$$p(x) = q(x)(x^2+1) + (a+bx).$$

Reason: I can divide $p(x)$ by (x^2+1) and get the quotient $q(x)$ & remainder $r(x)$ $\underbrace{}_{}$ must be degree $< \deg(x^2+1)$.

$$\mathbb{R}[x]/\langle x^2+1\rangle = \{a + bx + \langle x^2+1\rangle : a, b \in \mathbb{R}\}.$$

Addition is obvious $[a+bx] + [c+dx] = [(a+c) + (b+d)x]$.

Mult: $[a+bx][c+dx] = [ac + (ad+bc)x + bdx^2]$

$$= [ac + (ad+bc)x + bdx^2 - \underbrace{bd(x^2+1)}_{\in \langle x^2+1\rangle}]$$

$$= [(ac-bd) + (ad+bc)x]$$

$$(A+Bi)(C+Di) = (AC-BD) + (BC+AD)i$$

$\therefore$ $\mathbb{R}[x]/\langle x^2+1\rangle \cong \mathbb{C}$.

with $[a+bx] \longmapsto a+bi \in \mathbb{C}$

Let's compute $\mathbb{Z}[x]/\langle x\rangle$.

$$\langle x\rangle = \{x\,p(x) : p(x)\ \mathbb{Z}[x]\}$$

$$= \{a_1x + a_2x^2 + \cdots + a_kx^k : a_j \in \mathbb{Z}\ \forall j\}$$

$$= \{a_0 + a_1x + a_2x^2 + \dots a_kx^k + \langle x\rangle : a_j \in \mathbb{Z}\ \forall j\}$$

$$= \{a + \langle x\rangle : a \in \mathbb{Z}\}.$$

$a + \langle x\rangle \mapsto a$ $\cong$

$$\cong \mathbb{Z}.$$

integral domain, not a field.

$\Rightarrow$ $\langle x\rangle$ is prime, but not maximal.

(We can see also $\langle x\rangle \subsetneq \langle 3, x\rangle \subsetneq \mathbb{Z}[x]$)